

Risk Adjustment for Poker Players

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Introduction

In this article we consider risk aversion for advantage poker players. We discuss a sense in which poker players can proportionally Kelly-bet. This results in a method of prescribing bankroll requirements depending on risk aversion. It also provides a companion to the results of [CI] (in this volume) whose formulæ will apply to poker players exercising bankroll management and stake selection as prescribed. Employing a continuous diffusion model as in [CI], we show that if stakes are chosen in accordance with bankroll as prescribed, then the bankroll will follow the same stochastic process as for a fractional Kelly bettor with unit variance. One's risk-aversion can be parameterized by an instantaneous risk of ruin, or equivalently by a Kelly fraction k , which in turn corresponds to a utility function. The Kelly fractions we recommend are ones commonly used by blackjack professionals.

In the second section we use the concept of Certainty Equivalent (CE) to quantify how big a pot must be in order to call a bet with a drawing hand, depending on risk aversion. This makes precise a common assertion that one should forgo borderline positive expectation wagers that have high variance. As one might expect, except for longshot draws and low bankroll situations, the effect of risk-aversion is small but not negligible. We solve for the CE break-even points exactly for a range of Kelly fractions and provide numerical computations. Benefitting from the brevity of the one-half Kelly CE break-even formula, we obtain a simplified formula to approximate CE break-even points, in the last subsection.

Our investigation is based on limit holdem. With sufficient data the same considerations should apply to other wagering, such as no-limit holdem

and tournaments. The excess pot odds values are analogous to risk-averse playing indices in blackjack.

1 Proportional Betting for Poker Players

A common rough benchmark for limit holdem players (at least up to mid-stakes) is an hourly expectation of 1 big bet and a variance of 100 big bets squared. Thus the unitless ratio of hourly expectation to hourly standard deviation is 10. While your mileage may vary, we shall see that for our analysis, the *ratios* of expectation to variance and expectation to standard deviation are what matter. Some games may have higher expectation, but often their variance is higher, and the reverse may occur. We model the benchmark situation as Brownian motion with linear drift $r = 1$ (big bets/hour) and hourly standard deviation $s = 10$ big bets.

Poker players do not resize their bets as they do in blackjack. But they essentially choose their betting levels by which game they play. Let us assume a fixed ratio of expectation to standard deviation equal to a positive constant r/s . We model this game as Brownian motion with linear drift as earlier in this section. The resulting risk of ruin is given in [CI, 3.3 Corollary 3] as

$$\exp(-2xr/s^2)$$

where x is the bankroll. Now let us assume that the player's risk tolerance is specified by her Kelly fraction k , and their attendant risk of ruin $\exp(-2/k)$. Setting the two risks of ruin equal, we get

$$xr/s^2 = 1/k$$

i.e.

$$x = s^2/kr$$

This says that the k times Kelly player ideally has a bankroll always equal to s^2/kr .

We note for our benchmark ($s^2/r = 100$) game:

An optimal (full Kelly) player ideally always choose a game where the bankroll is 100 big bets, and this gives the optimal geometric growth rate.

As we noted in [CI, section 1], betting with $k > 1$ is always suboptimal. *Therefore is always wrong to have less than s^2/r (=100 big bets in benchmark) as a bankroll.* A $\frac{1}{3}$ -Kelly player will have 300 big bets. The more conservative quarter-Kelly players need 400 big bets, etc. This gives an answer to bankroll requirements for holdem players in terms of fractional Kelly betting. The range of answers seem be higher than bankroll recommendations commonly given (sometimes by erroneous reasoning) by poker experts, e.g. [Mal], www.twoplustwo.com.

Overall, more conservative money management is recommended. Important reasons for scaling back k for poker versus blackjack include the relative lack of certainty about r and s . Using a value a $k = 1/6$ and a bank of 600 big bets would not seem unreasonable. As we pointed out above, even smaller values of k may be appropriate. For an individual, the bankroll is one's total net worth minus expenses (including the present value of future earnings). However, many defy this definition and play with a "gambling bankroll" or "what they can afford to lose". In such artificial cases, playing closer to the optimal $k = 1$ may be recommended.

As a practical matter, suppose one settles at a risk tolerance of say around $k = 1/6$ or $k = 1/5$ and is playing a game with 600 big bets. If a losing streak of 100 big bets occurs, then it is time to consider scaling back to a smaller game. Although the resizing is not perfect, one can operate within certain risk tolerance bounds.

One's bankroll B with constant rate of return r and standard deviation s (for a session of fixed length) is modeled by the stochastic equation

$$dB = rdt + sdW,$$

which results in Brownian motion with linear drift at rate r . Mathematically, the player that chooses stakes according to his or her bankroll as we have just prescribed is equivalent to a fractional Kelly bettor. Precisely, the equivalence is given by the following observation, whose proof is a simple algebraic simplification.

Conclusion 1 *Consider the equation $dB = rdt + sdW$ where W is a standard Wiener process. If $\mu = r/s$ is a positive constant and stakes are always chosen so that $B = s^2/(kr)$, then the equation is the same as $dB = k\mu^2Bdt + k\mu BdW$. The bankroll will thus follow the same process as the fractional Kelly bettor. The latter equation is our diffusion for k times Kelly, with unit, as discussed in section 1 in [CI].*

Proof. Set

$$\mu = \frac{r}{s}$$

and assume that this ratio is constant. A player who continuously adjusts stakes so that $B = \frac{s^2}{kr}$ satisfies

$$\begin{aligned}
 dB &= r \frac{B}{B} dt + s \frac{B}{B} dW \\
 &= r \left[\frac{kr}{s^2} \right] B dt + s \left[\frac{kr}{s^2} \right] B dW \\
 &= k \left(\frac{r}{s} \right)^2 B dt + k \frac{r}{s} B dW \\
 &= k \mu^2 B dt + k \mu B dW
 \end{aligned}$$

This is proportional betting paradigm for fractional Kelly betting, with growth rate $\mu = \frac{r}{s}$ and $\sigma = 1$. ■

2 Micro Risk Adjustment

A basic wager is to win a pot of p bets, risking a call of 1 unit where the drawing odds are $1 : d$, i.e., the wager will be won with probability $\frac{1}{d+1}$.

The expected value of the wager is

$$\frac{p}{d+1} - \frac{d}{d+1} = \frac{p-d}{d+1}.$$

Let $x = p - d$ the *excess pot odds*. It is rudimentary that $p = d$ (i.e. $x = 0$) is the break-even point for expectation.

The *Certainty Equivalent* (CE) of a wager X is the risk-free value that has utility equal to the expected utility of X . CE is used as a risk-adjusted way of comparing bets, depending on the choice of utility function $u(t)$. We use utility functions

$$u(t) = \frac{t^{1-\frac{1}{k}}}{1-\frac{1}{k}}$$

which correspond to the Kelly fraction k with $0 < k < 1$. Full Kelly betting at $k \rightarrow 1$ corresponds to $u(t) = \ln t$.

We CE equal to zero and solve for x using various values of d , k and B . It is important to keep in mind that the values of B and x are measured in units of “bets”. So if when the (limit holdem) pot is raised your bankroll is numerically halved. The values of x give the excess pot size required to break even in CE, in excess of what would be required by pure expectation. Thus, to break even in CE the pot size needs to be $p = x + d$.

2.1 $k = 1$

The CE break-even point for logarithmic utility is expressed by

$$v(u(B-1)\frac{d}{d+1} + u(B+x+d)(\frac{1}{d+1})) = B$$

where $u(t) = \ln t$ and $v(t) = \exp t$, which has exact solution for x :

$$f(d, B) = x = B \exp(-(\ln(B-1)d + (\ln B)d) - B - d)$$

2.2 $k = .5$

We set $u(t) = -t^{-1} = v(t)$ and obtain the relatively simple CE break-even point function

$$f(x, B) = \frac{d(d+1)}{B-d-1}.$$

2.3 $k = 1/3$

Let $u(t) = -1/t^2$ and $v(t) = (-t)^{-\frac{1}{2}}$. The meaningful solution to the quadratic break-even equation is

$$x = \frac{1}{2(1+d-2Bd+B^2-2B)} \cdot \frac{(2Bd-2d-2B-2d^2+2dB^2+4Bd^2+4B^2-2B^3)}{-2\sqrt{(B^2+dB^2-4B^3+5dB^4-4B^3d+6B^4-2B^5d-4B^5+B^6)}}.$$

2.4 $k = 1/4$

Let $u(t) = -1/t^3$ and $v(t) = (-t)^{-\frac{1}{3}}$. The meaningful solution to the cubic break-even equation is

$$x = \frac{B}{d+1+3B^2d-3dB-B^3+3B^2-3B} \cdot \left((-B^3+3B^2-3B+1)(d+1+3B^2d-3dB-B^3+3B^2-3B)^2 \right)^{\frac{1}{3}} - B - d$$

2.5 Break-even tables with fixed k

Using the break-even formulae above we tabulate values of x . The values of d are chosen to reflect some holdem drawing odds. E.g. $d = 46$ is a longshot draw to a single out, whereas $d = 4$ is approximately for a straight or flush draw.

$k = 1$

d	$B = 25$	$B = 50$	$B = 100$	$B = 200$
4	.435	.208	.102	.051
7	1.27	.596	.289	.142
11	3.17	1.44	.690	.338
16	7.04	3.08	1.45	.701
22	14.4	5.98	2.75	1.32
46	92.5	30.6	12.8	5.87

$k = 1/2$

d	$B = 50$	$B = 100$	$B = 200$	$B = 400$
4	.444	.211	.103	.051
7	1.3	.609	.292	.143
11	3.47	1.5	.702	.340
16	8.24	3.28	1.49	.712
22	18.7	6.57	2.89	1.34
46	721	40.8	14.1	6.13

$k = 1/3$

d	$B = 75$	$B = 150$	$B = 300$	$B = 600$
4	.448	.211	.103	.0506
7	1.36	.613	.293	.143
11	3.60	1.52	.707	.341
16	8.81	3.35	1.50	.713
22	21.4	6.81	2.90	1.35
46	—	47.2	14.7	6.22

$k = 1/4$				
d	$B = 100$	$B = 200$	$B = 400$	$B = 800$
4	.450	.212	.103	.051
7	1.37	.616	.292	.143
11	3.67	1.53	.709	.342
16	9.14	3.39	1.51	.715
22	23.2	6.94	2.91	1.36
46	—	51.8	15.0	6.27

The missing values do not exist as real numbers, so unless the pot has imaginary chips, you should not call in these situations. The paradoxical lack of real solutions is explained by the fact that the utility function is bounded above.

Notice that the values for fixed kB (e.g. $kB = 100$ in boldface) are very close for a range of values of B and d . This is perhaps unsurprising given the popular continuous CE approximation $E - \frac{s^2}{2kB}$ with expectation $E = \frac{p-d}{d+1}$, though it is an approximation that we have discarded as inaccurate for the current computations.

2.6 Approximations

We use $k = .5$ as a model and adjust for other k 's proportionally

$$x = \frac{d(d+1)}{2k(B-d-1)}$$

or as a slightly more crude underestimate:

$$x = \frac{d(d+1)}{2kB}$$

For the sake of comparison, we show the approximated values versus the more crudely underestimated ones for two values of kB . The result is that the approximated values are quite good except for the longshot draws.

$kB = 50$					
d	$k = 1$	$k = .5$	$k = 1/3$	$k = 1/4$	$\frac{d(d+1)}{2kB}$
4	.208	.210	.211	.212	.20
7	.595	.609	.613	.616	.56
11	1.44	1.50	1.52	1.53	1.3
16	3.08	3.28	3.35	3.39	2.72
22	5.98	6.57	6.81	6.94	5.06
46	30.6	40.8	47.2	51.8	21.6

$kB = 100$					
d	$k = 1$	$k = .5$	$k = \frac{1}{3}$	$k = \frac{1}{4}$	$\frac{d(d+1)}{200}$
4	.102	.103	.103	.103	.10
7	.289	.292	.293	.293	.28
11	.689	.702	.706	.709	.66
16	1.45	1.49	1.50	1.51	1.4
22	2.75	2.86	2.90	2.92	2.5
46	12.8	14.1	14.7	15.0	11

References

- [CI] W. Chin and M. Ingenoso, Risk Formulæ for Proportional Betting.
- [Mal] M. Malmuth, *Gambling Theory and Other Topics*, Two Plus Two Publishing Las Vegas 1999. *Poker Essays*, Two Plus Two Publishing 1996.